Lecture Notes for PHYS 4146

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The mean orbital radius of Mercury is about $57.9 \times 10^6 km$ *and its period is roughly 0.241 years. Remembering that GM for the sun is 1.477 km, show that the predicted perihelion shift of* $6\pi GM/r_c$ *radians per orbit corresponds to roughly 43 arc-seconds/century. Note: 1 arc-second =* $1^\circ/3600$*. We will assume that* r_c *is equivalent to the mean orbital radius although that is an approximation.*

25 Chapter 9: Orbits in Schwarzschild as Tests of GR

25.1 Light Ray Orbits

- We have two Killing vectors in Schwarzschild that also exist along a null geodesic

$$\begin{aligned} e &\equiv -\xi \cdot \mathbf{u} = (1 - 2M/r)\frac{dt}{d\lambda} \\ \ell &\equiv \eta \cdot \mathbf{u} = r^2 sin^2\theta \frac{d\phi}{d\lambda} \end{aligned}$$

- We now have a well-defined ratio of ℓ and e

$$b \equiv \frac{\ell}{e} = \frac{r^2(d\phi/d\lambda)}{(1-2M/r)(dt/d\lambda)} = r^2\left(1 - \frac{2M}{r}\right)^{-1}\frac{d\phi}{dt}$$

  which is a conserved quantity along any geodesic, including a photon's geodesic. It is the impact parameter of the photon's trajectory around the source of curvature.

- Third integral from

$$\mathbf{u}\cdot\mathbf{u} = g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$$

- The effective potential for photon orbits follows from the work on particle orbits leading to

$$g_{tt}\left(\frac{dt}{d\lambda}\right)^2 + g_{rr}\left(\frac{dr}{d\lambda}\right)^2 + \cancel{g_{\theta\theta}\left(\frac{d\theta}{d\lambda}\right)^2} + g_{\phi\phi}\left(\frac{d\phi}{d\lambda}\right)^2 = 0$$

$$-\left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-2}e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + r^2\frac{\ell^2}{r^4} = 0$$

$$-\left(1-\frac{2M}{r}\right)^{-1}e^2 + \left(1-\frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2 + \frac{\ell^2}{r^2} = 0$$

$$\frac{dW}{dr} = 0 = \frac{d}{dr}\left(\frac{1}{r^2} - \frac{2M}{r^3}\right) = -\frac{2}{r^3} - \frac{(-3)2M}{r^4}$$

$$\frac{1}{b^2} = \frac{1}{\ell^2}\left(\frac{dr}{d\lambda}\right)^2 + W_{eff}$$

$$= -\frac{2}{r^3} + \frac{6M}{r^4}$$

where $b \equiv \ell^2/e^2$, playing the role of $\mathcal{E}$, and the effective potential is

$$\frac{2}{r^3} = \frac{6M}{r^4}$$

$$W_{eff}(r) \equiv \frac{1}{r^2}(1 - 2M/r)$$

$$2r = 6M \quad \therefore \quad r = 3M$$

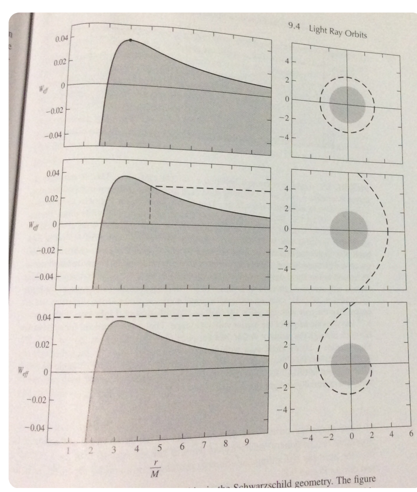
- The sign of ℓ indicates which direction the light ray is going around the center of attraction.

Class

find W_{eff} max

- The shape of these orbits: the maximum of $W_{eff}(r)$ occurs at $r = 3M$,

$$W_{eff}(3M) = \frac{1}{27M^2}\,.$$

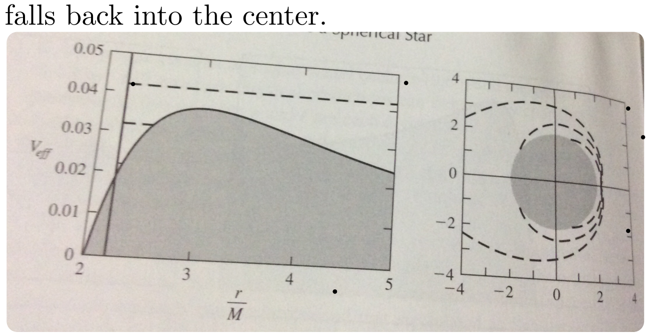


- Circular orbits of radius 3M are possible if $b^2 = 27M^2$, but they are unstable.

Since $W_{eff}(3M) = \frac{1}{27M^2}$ + circular $\Rightarrow W_{eff} = \mathcal{E} = \frac{1}{b^2}$

$b^2 = 27M^2$ but unstable!

- This means a light ray can not follow a circular orbit around the sun (3M = 4.5km) but there can exist one outside of a black hole.
- The shape of the orbit depends on $1/b^2$ being greater or less than W_{eff}
- Another set if trajectories occur between r=2M and r=3M. If $1/b^2 > 1/(27M^2)$ the light ray escapes, other wise it falls back into the center.



25.2 Light Ray Orbits

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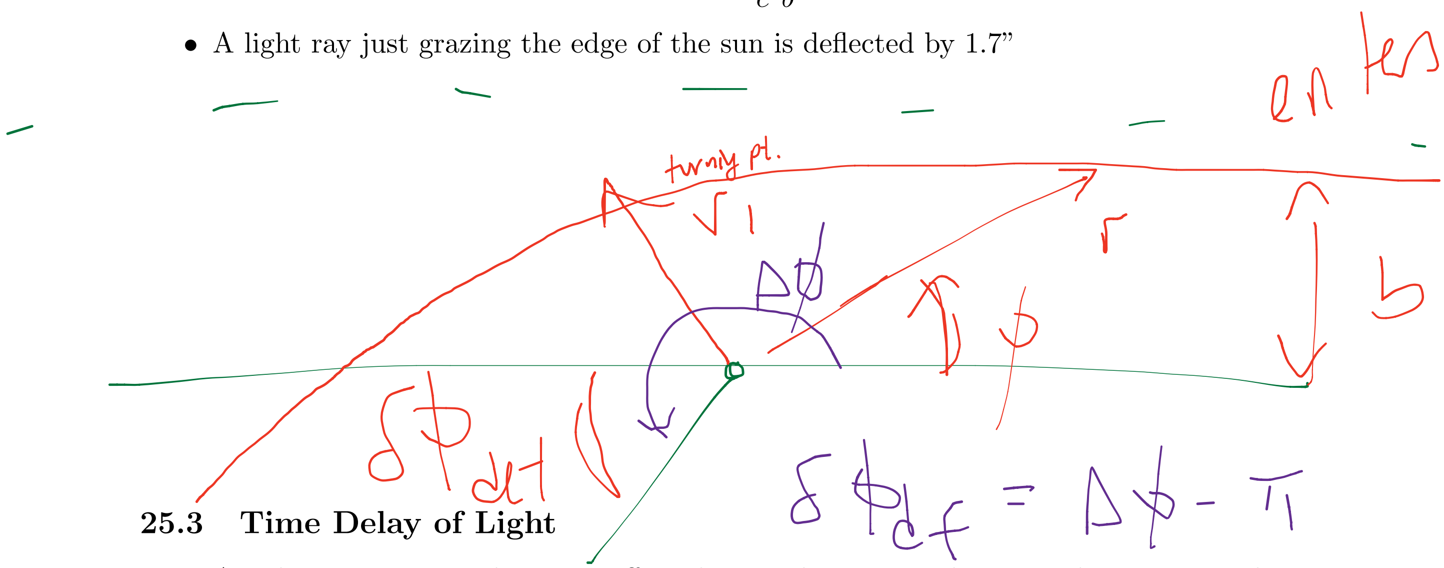
Light Deflection

- All material bodies will bend light trajectories somewhat.
- The deflection of light be the sun is one of the most important experimental tests of GR.
- It is also the mechanism behind gravitational lensing.
- The relativistic deflection of light when M/b is small can be written as

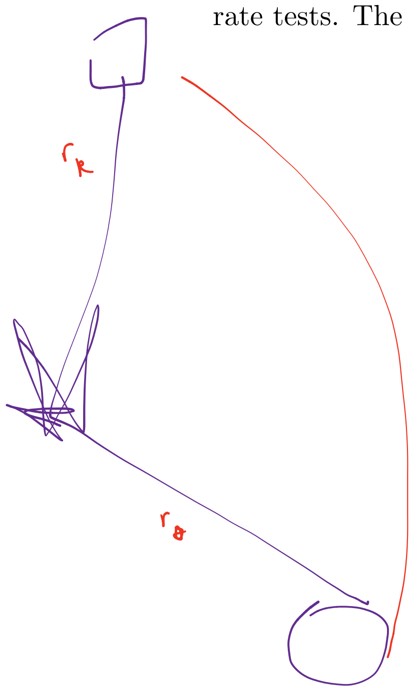
$$\delta\phi_{def}\frac{4GM}{c^2b}\,.$$

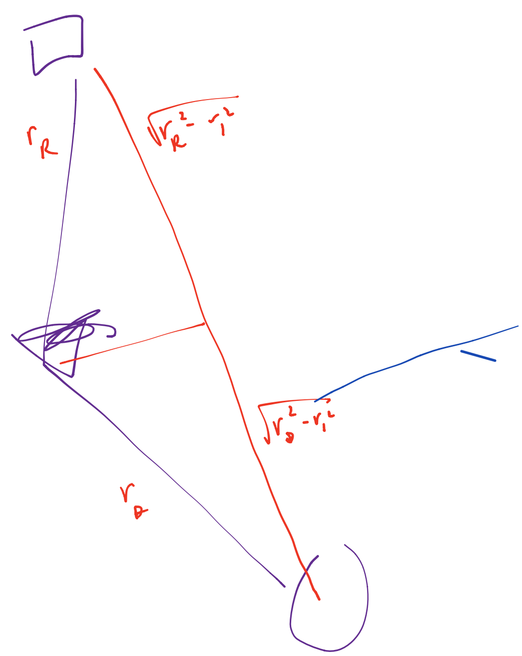
- A light ray just grazing the edge of the sun is deflected by 1.7”



25.3 Time Delay of Light

- Another interesting relativistic effect that can be measured in our solar system is the time delay of light passing near the sun.
- The tests for this are called radar-ranging techniques are are some of the more accurate tests. The effect is called the Shapiro time delay.





$$\frac{dt/d\lambda}{dr/d\lambda} = \frac{e(1-2M/r)^{-1}}{\pm\sqrt{\frac{\ell^2}{b^2} - \frac{\ell^2}{b^2} W_{eff}}} = \pm\frac{1}{b}\left(1-\frac{2M}{r}\right)^{-1}\left[\frac{1}{b^2} - W_{eff}\right]^{-1/2}$$

- Treating the Earth as stationary through this experiment, we can predict the total time between emission and reception of the light signal measured by a clock on the Earth - this will be the Schwarzschild coordinate time.
- $(\Delta t)_{total}$ is calculate the same way we as orbits, using $dt/d\lambda$ and $dr/d\lambda$.

$$\frac{dt}{dr} = \pm\frac{1}{b}\left(1 - 2M/r\right)^{-1}\left[\frac{1}{b^2} - W_{eff}(r)\right]^{-1/2}$$

  The total elapsed time is

$$(\Delta t)_{total} = 2t(r_\oplus, r_1) + 2t(r_R, r_1)$$

- This leads us to the excess time spent due to the curvature in the Schwarzschild geometry

$$(\Delta t)_{excess} \equiv (\Delta t)_{total} - 2\sqrt{r_\oplus^2 - r_1^2} - 2\sqrt{r_R^2 - r_1^2}\,.$$

- The largest effect occurs when r_1 is closest to the solar radius meaning that $r_1/r_\oplus << 1$ and $r_1/r_R << 1$

$$(\Delta t)_{excess} \approx \frac{4GM}{c^3}\left[log\left(\frac{4r_R r_\oplus}{r_1^2}\right) + 1\right]$$

25.4 Bound Orbits for Particles in Schwarzscild Geomtrey

What is their shape: $r(\phi)$ or $\phi(r)$

- $\mathcal{E} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}$ solve for $\frac{dr}{d\tau}$ and get

$$\frac{dr}{d\tau} = \pm\sqrt{2(\mathcal{E} - V_{eff})}$$

- $\ell = r^2 sin\theta\frac{d\theta}{d\tau}$ which, for $\theta = \pi/2$ give

$$\frac{d\phi}{d\tau} = \frac{\ell}{r^2}$$

- This leads to

$$\frac{d\phi}{dr} = \frac{d\phi/d\tau}{dr/d\tau} = \pm\frac{\ell}{r^2\sqrt{2(\mathcal{E} - V_{eff})}}$$

  where the sign corresponds to the direction in ϕ the particle moves in increasing r.
- Integrating will get the geodesic but not all that illuminating so let's do a special property - precession!

- Precession: if an orbit does not close, it processes.
- orbit: passage between 2 successive turning points
- orbits close: if magnitude of the angle that is swept out by the orbit $\Delta\phi - 2\pi = 0$.
- precess angle: $\delta\phi_{prec} = \Delta\phi - 2\pi$

- Planet around a star moves a minimum radius, r_1, to a maximum radius, r_2. Unlike the elliptic orbits of Kepler, the orbit does not close. The angular position in closest approach advances sightly at each return (precession of perihelion).
- $$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2\sqrt{2(\mathcal{E} - V_{eff})}}$$

  plugging in

  $$V_{eff}(r) \equiv \frac{1}{2}\left[\left(1 - \frac{2M}{r}\right)\left(1 + \frac{\ell^2}{r^2}\right) - 1\right] = -\frac{M}{r} + \frac{\ell^2}{2r^2} - \frac{M\ell^2}{r^3}$$

  and $\mathcal{E} = \frac{e^2-1}{2}$ we get

  $$\frac{d\phi}{dr} = \pm\frac{\ell}{r^2}\left[e^2 - (1 - 2M/r)(1 + \ell^2/r^2)\right]^{-1/2}$$
  $$\Delta\phi = 2\ell\int_{r_1}^{r_2}\frac{dr}{r^2}\left[e^2 - (1 - 2M/r)(1 + \ell^2/r^2\right]^{-1/2}$$

  where the 2 is coming from the precession angle being twice the angle swept out between r_1 and r_2.

- To 1st order in $1/c^2$

$$\Delta\phi = 2\pi + 6\pi \left(\frac{GM}{c\ell}\right)^2$$

which leads us to an expression for the precession

$$\delta\phi_{prec} = 6\pi \left(\frac{GM}{c\ell}\right)^2 + \mathcal{O}\left(\frac{1}{c^2}\right) \tag{1}$$

- To this accuracy, use Newtonian orbits to estimate $\ell(\epsilon, a)$ where ϵ is the eccentricity of the orbit and a is its semi-major axis such that

$$\begin{aligned} L^2 &\approx \mu k s(1-\epsilon^2), \text{ where } \mu \approx m \text{ and } k \approx GmM \\ \ell^2 &= \frac{L^2}{m^2} = GMa(1-\epsilon^2) \end{aligned}$$

which gives us

$$\delta\phi_{prec} = \frac{6\pi G}{c^2}\frac{M}{a(1-\epsilon^2)}$$

the first order relativistic precession of the perihelion. The planet with the largest $1/a$ will have the largest effect (i.e Mercury).

- Mercury precesses by $43''$/ century.